# SUGGESTED SOLUTION TO HOMEWORK 5

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**Problem 1.** Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ a one-to-one bounded linear operator. Show that $T^{-1} : T(X) \to X$ is bounded if and only if $T(X)$ is closed.

*Proof.* $\Rightarrow$:Suppose $T^{-1} : T(X) \to X$ is bounded. For arbitrary $\{y_n\}_{n\geq 1} \subset T(X)$ which converges to $y \in Y$, we claim that there exists $x \in X$ such that $y = T(x)$. Indeed, we define $x_n = T^{-1}(y_n)$, then

$$\|x_n - x_m\| \leq \|T^{-1}\|\|y_n - y_m\|,$$

which implies that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $X$, since $X$ is a Banach space, there exists $x \in X$ such that $\{x_n\}_{n\geq 1}$ converges to $x$. Moreover, since for arbitrary $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\|y_n - y\| < \varepsilon,$$

and there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$,

$$\|x_n - x\| < \|T\|^{-1}\varepsilon,$$

then for all $n > \max\{N_1, N_2\}$,

$$\begin{aligned}\|y - T(x)\| &\leq \|y - y_n\| + \|T(x_n) - T(x)\| \\ &\leq \|y - y_n\| + \|T\|\|x_n - x\| \\ &< 2\varepsilon,\end{aligned}$$

which implies that $y = T(x)$ by the arbitrariness of $\varepsilon$.

$\Leftarrow$: Suppose $T(X)$ is closed, since $Y$ is a Banach space and $T(X) \subset Y$ is a subspace, we have $T(X)$ is also a Banach space. Since $T : X \to Y$ is a injection, therefore $T : X \to T(X)$ is a bijection, then we have $T^{-1} : T(X) \to X$ is a bounded linear operator by the inverse mapping theorem. □

**Problem 2.** Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces. We define a norm $\|\cdot\|$ on the vector space $X \times Y$ by

$$\|(x, y)\| = \max\{\|x\|_1, \|y\|_2\}.$$

(a) Show that $\|\cdot\|$ is indeed a norm on $X \times Y$ and the topology defined by this norm coincides with the product topology on $X \times Y$.

(b) Prove that if $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are Banach space, then $(X \times Y, \|\cdot\|)$ is a Banach space.

(c) Show that if $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are Banach space and $T : X \to Y$ is closed, then the Graph $G(T) = \{(x, Tx) : x \in X\}$ is a Banach space.

*Proof.* (a) We prove $\|\cdot\|$ is a norm. It is clear that $\|\cdot\|$ satisfies homogneity and $\|(0, 0)\| = 0$, moreover, if $\|(x, y)\| = 0$, then

$$\|x\|_1 \leq \|(x, y)\|, \quad \|y\|_2 \leq \|(x, y)\|,$$

which implies that $x = 0$ and $y = 0$, therefore $(x, y) = (0, 0)$. It suffices to prove that $\|\cdot\|$ satisfies the triangle inequality. Indeed, for arbitrary $(x_1, y_1)$ and $(x_2, y_2)$,

$$\begin{aligned}\|(x_1 + x_2, y_1 + y_2)\| &\leq \max\{\|x_1 + x_2\|_1, \|y_1 + y_2\|_2\} \\ &\leq \max\{\|x_1\|_1 + \|x_2\|_1, \|y_1\|_2 + \|y_2\|_2\} \\ &\leq \max\{\|x_1\|_1, \|y_1\|_2\} + \max\{\|x_2\|_1, \|y_2\|_2\} \\ &\leq \|(x_1, y_1)\| + \|(x_2, y_2)\|.\end{aligned}$$

We also claim that the topology defined by $\|\cdot\|$ coincides with the product topology on $X \times Y$. Indeed, on the one hand, let $U$ and $V$ be two open sets in $X$ and $Y$ respectively, then for arbitrary $p := (x, y) \in U \times V$, since $x \in U$ and $y \in V$, there exist $r_1 > 0$ and $r_2 > 0$ such that $B_X(x, r_1) := \{x' \in X : \|x' - x\|_1 < r_1\} \subset U$ and $B_Y(y, r_2) := \{y' \in Y : \|y' - y\|_2 < r_2\} \subset V$, therefore for $r = \min\{r_1, r_2\}$, we have $B_{X\times Y}(p, r) := \{p' \in X \times Y : \|p' - p\| < r\} \subset U \times Y$, which implies that $U \times V$ is an open set under the topology induced by $\|\cdot\|$.

On the other hand, let $O \subset X \times Y$ be an open set under the topology induced by $\|\cdot\|$, for arbitrary $p \in O$, there exists $r > 0$ such that $B_{X\times Y}(p, r) := \{p' \in X \times Y : \|p' - p\| < r\} \subset O$, then $B_X(x, r) \times B_Y(y, r) \subset B_{X\times Y}(p, r) \subset O$, which implies $p$ is a interior point under the product topology. By the arbitrariness of $p \in O$, we have $O$ is also an open set under the product topology of $X \times Y$.

(b) Let $\{p_n := (x_n, y_n)\}_{n\geq 1}$ be a Cauchy sequence in $X \times Y$, then for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m > N$,

$$\|p_n - p_m\| < \varepsilon,$$

which implies that

$$\|x_n - x_m\|_1 < \varepsilon, \quad \|y_n - y_m\| < \varepsilon,$$

which implies that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are Cauchy sequence in $X$ and $Y$ respectively. Since $X$ and $Y$ are Banach space, there exists $x \in X$ and $y \in Y$ such that there exist $N_1$ and $N_2$ such that for all $n > N_1$ and $n > N_2$,

$$\|x_n - x\|_1 < \varepsilon, \quad \|y_n - y\|_2 < \varepsilon.$$

Denote $p := (x, y)$, therefore for $n > \max\{N_1, N_2\}$,

$$\|p_n - p\| = \max\{\|x_n - x\|_1, \|y_n - y\|_2\} < \varepsilon,$$

which implies that $p$ is a limit of $\{p_n\}_{n\geq 1}$.

(c) Since $T$ is closed, therefore $G(T)$ is a closed subspace in $X \times Y$. Since $X \times Y$ is a Banach space, therefore $G(T)$ is also a Banach space. □

**Problem 3.** Show that the space

$$Y = \{x \in C^1[0, 1] : x(0) = 0\}$$

equipped with the sup-norm is not a Banach space.

*Proof.* For $n \in \mathbb{N}$, consider

$$x_n(t) = \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{n}} - \sqrt{\frac{1}{4} + \frac{1}{n}},$$

then it is clear that $x_n \in Y$. We claim that $\{x_n\}_{n\geq 1}$ converges to the function $x(t) = \left|t - \frac{1}{2}\right| - \frac{1}{2}$ in the sup-norm. Indeed, by the triangle inequality,

$$|x_n(t) - x(t)| \leq \left| \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{n}} - \left|t - \frac{1}{2}\right| \right| + \left| \sqrt{\frac{1}{4} + \frac{1}{n}} - \frac{1}{2} \right| \leq \frac{2\sqrt{n}}{n},$$

therefore for aribtrary $\varepsilon > 0$ and $n > 4\varepsilon^{-2}$, we have

$$\sup_{t\in[0,1]} |x_n(t) - x(t)| < \varepsilon,$$

for all $t \in [0, 1]$. However $x \notin Y$, which implies that $Y$ is not a Banach space. □

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